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Determinant-gravity: cosmological implications

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Abstract: We analyze the action $\int d^4x \sqrt{\det ||\mathcal{B}\, g_{\mu\nu} + \mathcal{C}\, R_{\mu\nu}||}$ as a possible alternative or addition to the Einstein gravity. Choosing a particular form of $\mathcal{B}(R) = \sqrt{R}$ we can restore the Einstein gravity and, if $\mathcal{B} = m^2$, we obtain the cosmological constant term. Taking $\mathcal{B} = m^2 + \mathcal{B}_1 R$ and expanding the action in $1/m^2$, we obtain as a leading term the Einstein lagrangian with a cosmological constant proportional to m^4 and a series of higher order operators. In general case of non-vanishing $\mathcal{B}$ and $\mathcal{C}$ new cosmological solutions for the Robertson-Walker metric are obtained.

Keywords: Classical Theories of Gravity, Cosmology of Theories beyond the SM.

There are numerous suggestions in the literature for modification of the classical Einstein action of general relativity:

$$S_E = \frac{m_{Pl}^2}{16\pi}\int d^4x\sqrt{g}\,(R-2\Lambda) \tag{1}$$

where R is the curvature scalar, $g = -\det|g_{\mu\nu}|$ is the determinant of the metric tensor, Λ is the cosmological constant and m_{Pl} is the Planck mass. Majority of such attempts (at least in 4-dimensional space-time) are based on the same structure of the action integral with an addition to the Einstein term of some scalar functions of R and/or of combinations of the Ricci, $R_{\mu\nu}$, and/or Riemann, $R_{\mu\nu\alpha\beta}$, tensors ($\int d^4x\sqrt{g}\,\mathcal{L}(R, R_{\mu\nu}, R_{\mu\nu\alpha\beta})$). Usually, but not necessarily, one considers quadratic terms in the curvature proportional to R^2, $R_{\mu\nu}R^{\mu\nu}$, and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. Such terms do not introduce a scale dependent parameter or, in other words, they enter into the action with dimensionless coefficients. Some of such and higher order or even non-local terms may appear as a result of quantum corrections, see e.g. the book [1].

Such a form of the lagrangian density, i.e. a scalar function multiplied by the determinant of the metric tensor, is dictated by the demand of the invariance of the action with respect to general coordinate transformations. However, this is not the only way to ensure this invariance. In fact any scalar function multiplied by a determinant of a second rank tensor would also be invariant with respect to the choice of coordinates [2].

Already in 1934 Born and Infeld [3] proposed a new action for electromagnetism (reducing to the Maxwell action for small amplitudes) of the form

$$S_{BI} = \int d^4x\sqrt{\det||m^2\,g_{\mu\nu} + \mathcal{F}_{\mu\nu}||} \tag{2}$$

with $\mathcal{F}_{\mu\nu}$ the electromagnetic field strength.

A generalization of the above action to gravity was suggested by Deser and Gibbons in ref. [4], where they studied the physical criteria that such a theory should satisfy.

Along the same line we analyze maximally symmetric spaces where only two independent tensors $g_{\mu\nu}$ and $R_{\mu\nu}$ are needed to specify geometrical structure of space-time. Within such a hypothesis we can write the general covariant action as:

$$\mathcal{S}_{\det} = \int d^4x\sqrt{\det||\mathcal{G}_{\mu\nu}||} \tag{3}$$

where $\mathcal{G}_{\mu\nu} = \mathcal{B}\,g_{\mu\nu} + \mathcal{C}\,R_{\mu\nu}$.[1] Below we make a simple choice for the coefficients: $\mathcal{B} = \mathcal{B}(R) = m^2 + \mathcal{B}_1 R$ where m, at this stage,[2] is a new mass parameter, while $\mathcal{B}_1$ and $\mathcal{C}$ are some dimensionless constant coefficients. In general, $\mathcal{B}$ and $\mathcal{C}$ could be arbitrary scalar functions of geometrical tensors, R, $R_{\mu\nu}$, $R_{\mu\nu\alpha\beta}$, etc.

[1]Such an ensemble of actions is more restricted than the one proposed by Deser and Gibbons [4] and does not satisfy the condition of absence of ghosts in the linearized version [4, 8].

[2]The Planck mass scale enters the lagrangian only when we define the coupling between matter and gravity.

JHEP11(2004)062

Somewhat similar modification with $\mathcal{L} = \sqrt{\det ||R_{\mu\nu}||}$ was suggested by Eddington [5], as we have found out from ref. [6], but the latter was considered as a pure affine theory depending on connection $\Gamma^{\alpha}_{\mu\nu}$ and its derivatives only (a generalization of the above results in the same framework is given by Vollick [7]), while we prefer to remain in the frameworks of metric theory in the same spirit as ref. [4].

It is non-trivial task to derive equations of motion from action (3), but in the case of maximally symmetric space, considered here, this problem is greatly simplified. One can show that:

$$\begin{aligned}\frac{\delta \mathcal{S}_{\rm det}}{\delta\, g^{\mu\nu}} &= -\mathcal{B}(R)\mathcal{P}_{\mu\nu} + \mathcal{B}'\,\mathcal{P}\,R_{\mu\nu} + (g_{\mu\nu}D^2 - D_\mu D_\nu)\times \\ &\times(\mathcal{B}'\mathcal{P}) + \left(\frac{\mathcal{C}}{2}\right)(D^2\mathcal{P}_{\mu\nu} + g_{\mu\nu}D_\alpha D_\beta \mathcal{P}^{\alpha\beta} - D_\alpha D_\mu \mathcal{P}^{\alpha}_{\nu} - D_\alpha D_\nu \mathcal{P}^{\alpha}_{\mu}) = 0 \qquad (4)\end{aligned}$$

where $\mathcal{B}' = d\mathcal{B}/dR$, D_μ is the covariant derivative and $D^2 = g_{\mu\nu}D^\mu D^\nu$.

The second rank tensor $\mathcal{P}_{\mu\nu}$ is defined as

$$\mathcal{P}_{\mu\nu} = \frac{\sqrt{\det||\mathcal{G}_{\mu\nu}||}}{2\sqrt{g}}\,\mathcal{G}^{(-1)}_{\mu\nu} \qquad (5)$$

and $\mathcal{P} = g^{\mu\nu}\mathcal{P}_{\mu\nu}$.

Remember that, according to our hypothesis, any geometric second rank tensor, in particular, $\mathcal{P}_{\mu\nu}$ can be expressed through a linear combination of two basic tensors: $a_1\, g_{\mu\nu} + a_2\, R_{\mu\nu}$, where $a_{1,2}$ are scalar functions of various invariants ($R^n, \mathrm{Tr}[R\cdot\dots\cdot R]$).

We checked explicitly that eq. (4) satisfies, as it should, the transversality condition, $D^\mu(l.h.s.) = 0$.

The physical consequences of the suggested above gravitational action can be studied from different angles.

In the most conservative approach we may consider the new action as a higher order correction to the classical Einstein term and restrict the new interactions with phenomenological considerations.

In this case we have to put into the r.h.s. of the equation of motion (4), the usual general relativity term, $8\pi T_{\mu\nu}/m^2_{Pl} - G_{\mu\nu}$, where $T_{\mu\nu}$ is the energy-momentum tensor of matter and $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor. In the most radical approach, instead, we can work only with the new action as a fundamental dynamical source of the gravitational field and may even try to introduce matter fields under the signs of det and the square root (work in progress).

In such a perspective it is important, first of all, to find out the proper limiting connection to the classical approach, eq. (1), trying to reconstruct the Einstein General Relativity limit using a special form of functions $\mathcal{B}$ and $\mathcal{C}$ (see below).

The first problem can be solved if we take $\mathcal{B}(R) = m^2 + \mathcal{B}_1 R$, as we mentioned above, and expand the action (3) in powers of $1/m^2$

$$S_{\rm det} \approx \int d^4x\sqrt{g}\left[m^4 + \left(2\,\mathcal{B}_1 + \frac{\mathcal{C}}{2}\right)m^2\,R + \left(\mathcal{B}_1^2 + \frac{\mathcal{B}_1\mathcal{C}}{2}\right)R^2 + \frac{\mathcal{C}^2}{4\,[R]^2} + \cdots\right] \qquad (6)$$

where $[R]^2 = R_{\mu\nu}R^{\mu\nu}$. The first and second terms in the expression above correspond to the Einstein action with a cosmological constant (see eq. (1)) while the higher order terms are the corrections to be treated perturbatively. It is clear that, if we want to get rid of the cosmological constant, in the radical approach, we cannot send naively $m \to 0$ because we lose the classical limit, hence we have to take into account canceling matter effect (work in progress).[3]

We will study eq. (4) in homogeneous and isotropic space-time with the FRW metric:

$$ds^2 = dt^2 - a^2(t)dr^2\,. \tag{7}$$

As is well known it is sufficient in this case to consider only the time-time component of equations of motion.

The explicit general solution of the above equation is quite cumbersome (and above the length of our paper) so we will try to find some results in physically interesting simple limits.

- For example it is easy to see that a De Sitter phase ($a(t) = e^{h\,t}$) is a solution of eq. (4) of the general action (3) if

$$m^2\left(m^2 - 3(4\,\mathcal{B}_1 + \mathcal{C})\,h^2\right) = 0 \tag{8}$$

  and in order to have real h solutions, the parameters $\mathcal{B}_1$ and $\mathcal{C}$ should lie in certain bounds. In particular there are solutions with fixed $h^2 = m^2/(\sqrt{3}(\mathcal{C} + 4\mathcal{B}_1))$ if $\mathcal{C}+4\,\mathcal{B}_1 > 0$ and $m \neq 0$ but we found that they are instable under small perturbations. On the contrary, if $m = 0$, there are solutions with generic $h \neq 0$ (and we checked their stability for any $\mathcal{B}_1, \mathcal{C}$).

- The study of possible power law solutions of eqs. (4), $a(t) = t^n$, is easier in the scale invariant case where $m = 0$ and $\mathcal{B} = \mathcal{B}_1\,R$. Note that the new action containing $\det||\mathcal{B}_1\,R\,g_{\mu\nu} + \mathcal{C}\,R_{\mu\nu}||$ can be meaningful only for a definite sign of the determinant. In contrast to $\det||g_{\mu\nu}||$ the latter is not guaranteed from the general principles but the sign may be fixed on the solutions of equations of motion. In figure 1 we show the signature of the determinant versus the ratio $\mathcal{C}/\mathcal{B}_1$ and $n = 2/(3+3\,w)$. We see that a well defined signature in the interval $-1 < w < 1$ is realized only for $\mathcal{C}/\mathcal{B}_1 = -4,\,0$: the second solution is, in a sense, "trivial" corresponding to the well known quadratic R^2- lagrangian, $\mathcal{B}_1^2\int d^4x\sqrt{g}\,R^2$, while the first solutions generates the traceless tensor $\mathcal{G}_{\mu\nu} = \mathcal{B}_1(R\,g_{\mu\nu} - 4\,R_{\mu\nu})$ whose equations of motion are studied below.

  For the power law solutions, $a(t) = t^n$, we find the constraint

$$w = -\frac{1}{3} + \frac{8(3+x)}{3(2+x)(6+x)} \tag{9}$$

  where $n = 2/(3+3\,w)$, $x = \mathcal{C}/\mathcal{B}_1$.

[3] A naive cancellation mechanism consist in the addition to eq. (3) of the term $-\int d^4x\,m^4\sqrt{\det||g_{\mu\nu}||}$.

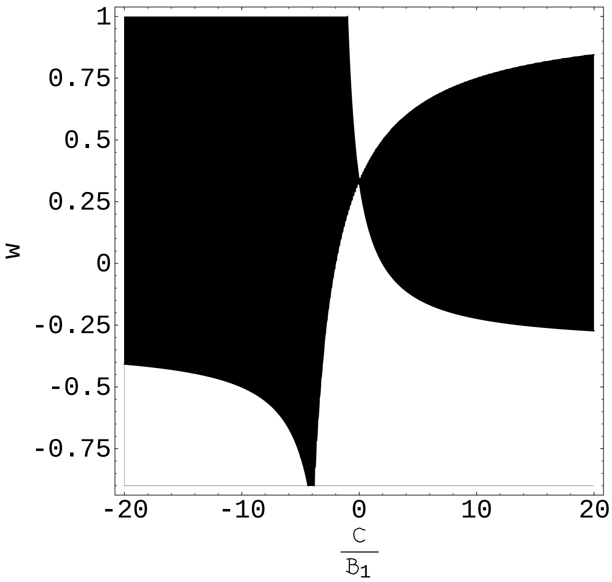

Figure 1: Signature of $\det ||\mathcal{B}_1\, R\, g_{\mu\nu} + \mathcal{C}\, R_{\mu\nu}||$ for $a(t) = t^n$ and $n = 2/(3+3w)$. The negative values are shown as black areas, while the positive ones are whites.

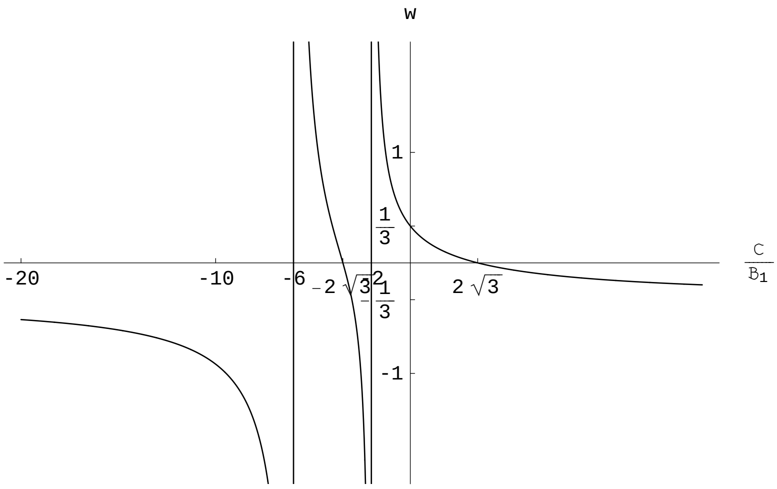

Figure 2: Plot of eq. (9) which gives the values of w as function of $\mathcal{C}/\mathcal{B}_1$ indicating when the power law $a(t) = t^{2/(3+3w)}$ ansatzs is a solution of the scale invariant lagrangian with $m = 0$.

This curve is showed in figure 2. In order to control stability of the solutions we have found that the first curve to the left corresponds to unstable solutions. For the central one there is a stability region for $0 \leq w < -1$ corresponding to $-2\sqrt{3} \leq x < 2(-3+\sqrt{3})$, and for the third curve to the right, stability is ensured if $-1/3 < w \leq 0$ and $x \geq 2\sqrt{3}$. Thus, there is no stable solution for $w > 0$ and the matter dominated regime ($w = 0$) is the stability border line.

For example the solution $a(t) = t^{1/2}$ corresponding to $w = 1/3$ (radiation dominated universe) are obtained for $x = -4$ and for $x = 0$ and both results are unstable.

JHEP11(2004)062

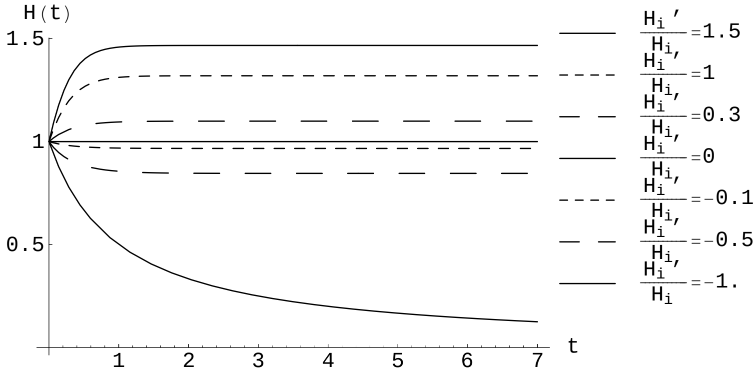

Figure 3: Solutions of eq. (11) in the case $H_i \geq 0$, where $H(t)$ is evolving for different initial conditions $H_i'/H_i = 1.5, 1, 0.3, 0, -0.1, -0.5, -1$ and $H_i = 1$.

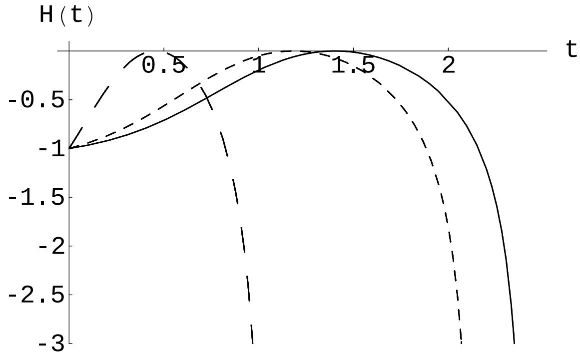

Figure 4: Solutions of eq. (11) in the case $H_i \leq 0$, where $H(t)$ is evolving for different initial conditions $H_i'/H_i =$ 0.3, 0.5 ,3 and $H_i = -1$.

- The minimal scale invariant case with $m = \mathcal{B}_1 = 0$

$$S_{\rm det} \to \mathcal{C}^2 \int d^4x \sqrt{\det ||R_{\mu\nu}||} \tag{10}$$

gives a tractable equation of motion:

$$\mathcal{C}^2\, H^4 \sqrt{\frac{3+\chi}{3(1+\chi)}} \left[\frac{\dot\chi}{H} \frac{2\chi^2 + 6\chi + 3}{(3+\chi)(1+\chi)} + 3\chi(1+\chi) \right] = 0 \tag{11}$$

where $H = \dot a / a$ is the Hubble parameter and $\chi = \dot H / H^2$.

At expansion regime, when $H > 0$ the solutions tend to the De Sitter one with $H \to const$ both for negative and positive initial values of $\dot H$ (see figure 3), while at contraction, when $H < 0$, the solutions may tend to singularity, $H \to -\infty$ if $\dot H_i < 0$ or to $H = 0$ if the initial value of $\dot H_i$ is positive (see figure 4). However, after reaching zero value. H does not change sign but becomes negative again and also reaches singularity. Thus, in this approximation singularity is not avoided, though the contraction may stop for a while.

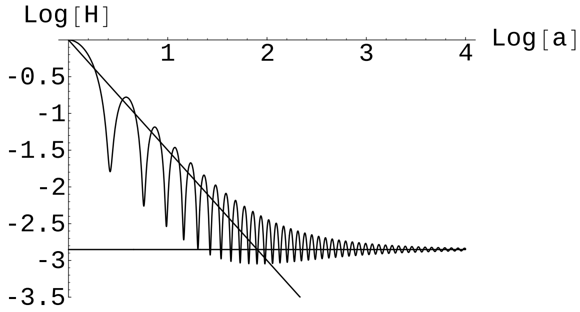

Figure 5: Solutions of eq. (13) in presence of matter and cosmological constant ($\rho = \rho_{matter} + \rho_\Lambda = e^{-3z} + 0.01$) ($z = \log[a(t)]$) in the case $H_i = 1$, $H_i' = 0$, $\kappa = 10$. We plot the value of $\log[H]$ and the descending straight line represents $\log[\rho_{matter}] = -3\,z$ while the horizontal one is the attraction point $3H^2 = \rho_\Lambda$.

- The scale invariant case with $m = 0$ and $\mathcal{C} = -4\,\mathcal{B}_1$ also deserves some comments. The action in this case is:

$$S_{\rm det} \to \mathcal{C}^2 \int d^4x \sqrt{\det ||R\,g_{\mu\nu} - 4\,R_{\mu\nu}||}\,. \tag{12}$$

The determinant of $\mathcal{G}_{\mu\nu}$ is sign definite and the ratio $\det \mathcal{G}_{\mu\nu} / \det g_{\mu\nu} = (2\dot{H})^4$. The equation of motion is very simple:

$$2\ddot{H}H + 6H^2\dot{H} - \dot{H}^2 = 0 \tag{13}$$

and can be integrated analytically in quadratures but numerical solution is faster and simpler. At expansion regime, when initially $H_i > 0$, the solution tends to a De Sitter one with $H \to const$. On contraction phase, when $H_i < 0$, we find that either H reaches negative and infinitely large value in finite time, if $\dot{H}_i < 0$, or H tends to zero, again in finite time if $\dot{H}_i > 0$.

Surprisingly equation (13) is the same that can be derived from the simple quadratic lagrangian $\int d^4x \sqrt{g}\,R^2$ [11] or, in our approach, from action (3) with $m = 0$ and $\mathcal{C} = 0$.

It is interesting to study eq. (13) in presence of the Einstein term and matter, that means we have to add the term $\kappa(T_{00} - G_{00})$ to the right hand side, where κ is a constant, $T_{00} = \rho$ is the energy density of matter, and $G_{00} = 3H^2$. Here we took the system of units with $m_{Pl}^2/8\pi = 1$.

In this case the behavior of the solution at expansion regime becomes different. If $\kappa > 0$ the solution $H(t)$ has an oscillating behavior. For vacuum energy, when $\rho = const$, the oscillations proceed around a constant average value (see figure 5), while for normal matter with $\rho \sim a^{-n}$, where $n = 3$ for non-relativistic matter and $n = 4$ for relativistic matter, the oscillations proceed around gradually decreasing average values (see figure 5). The acceleration parameter, $(\ddot{a}/a) = \dot{H} + H^2$ also oscillates and may be both negative and positive, depending upon time moment. The line $H = 0$ is a separation line of equation (13) and solutions can smoothly touch it but never cross.

JHEP11(2004)062

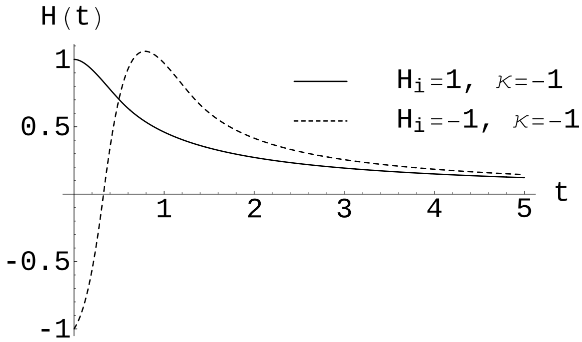

Figure 6: Solutions of eq. (14) in absence of matter ($\rho = 0$) and with the initial conditions $H_i = \pm 1$, $H_i' = 0, \kappa = -1$.

As we have already mentioned, the determinant of $\mathcal{G}_{\mu\nu}$ is sign definite only for $\mathcal{C}/\mathcal{B}_1 = 0, -4$. In other cases it may change sign in the course of evolution and the solution would encounter quite strange singularity. To avoid that we may try the action with the absolute value, $|\det \mathcal{G}_{\mu\nu}|$. It preserves invariance with respect to coordinate transformation, since the latter does not change the sign of determinant of a second rank tensor. Equations of motion in this case acquire an additional factor containing the sign function, $\epsilon[z] = z/|z|$. In particular the only change in eq. (4) is the change of $\mathcal{P}_{\mu\nu}$ by the absolute value under the sign of the square root.

As a result the singularity, at the point where the determinant vanishes, disappears and the equations of motions can be solved for all values of time. We have found some solutions of time-time component of the equations of motion for a particular case of $\mathcal{G}_{\mu\nu}$ coinciding with the Einstein tensor $\mathcal{G}_{\mu\nu} = \mathcal{C}(R_{\mu\nu} - g_{\mu\nu}R/2)$.

In this particular case, the equation for time-time component has a very simple form:

$$\ddot{H} = -3\dot{H}\,H + \kappa\sqrt{|2\dot{H} + 3H^2|}\,\epsilon[2\dot{H} + 3H^2]\frac{3H^2 - \rho}{H|H|} \tag{14}$$

where κ is a constant which may be both positive and negative, $\epsilon[z]$ is the sign function as has been already mentioned above. This sign function appears as a result of taking the absolute value of the determinant of $\mathcal{G}_{\mu\nu}$, while $|H|$ appears because this determinant is proportional to $\sqrt{H^2} = |H|$.

We have solved this equation for different special cases. At expansion regime, when $H > 0$, the solution without matter tends to $H = 2/3t$, i.e. to the matter dominated (MD) regime but without matter (see figure 6)).

The same behavior is found in presence of relativistic and non-relativistic matter. If vacuum energy is non-zero and initially sub-dominant then the solution starts from MD regime and then turns into De Sitter one with $H \to const$. An interesting feature of this solution is that before reaching an asymptotic constant value the solution oscillates around it with the frequency $\omega^2 \approx \kappa m_{Pl}^2/8\pi$ (see figure 7). Such fast oscillations would give rise to gravitational particle production with energies near the Planck energy. This process might be effective at the early inflationary stage and probably even now when $H^2 m_{Pl}^2$

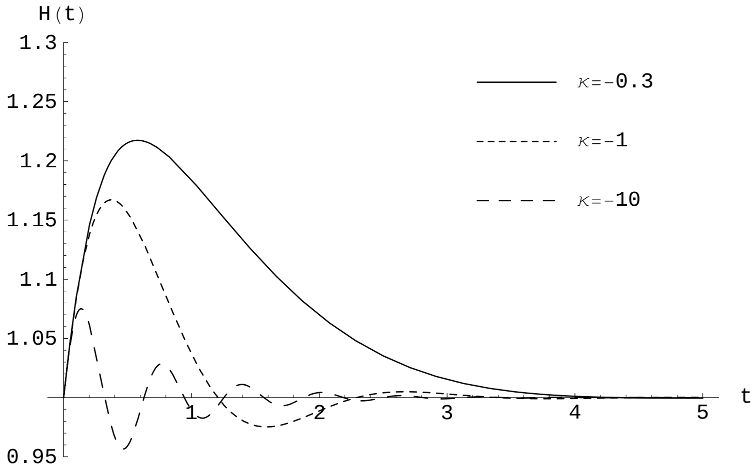

Figure 7: Solutions of eq. (14) in presence of the cosmological constant, $\rho_{vac} = 3$, and with the initial conditions $H_i = 1$, $H'_i = 1$ for different values of $\kappa = -0.3, -0.1, -10$.

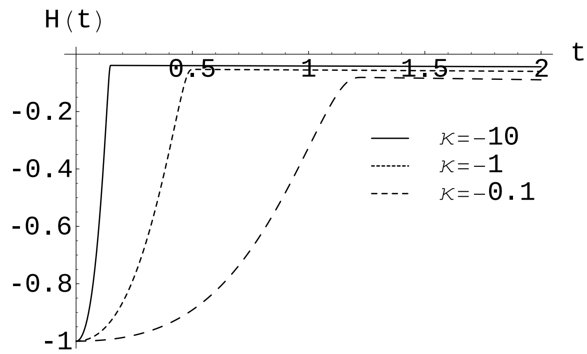

Figure 8: Solutions of eq. (14) in presence of matter density ($\rho = 0.1/a(t)^3$, $a(0) = 1$) with the initial conditions $H_i = -1$, $H'_i = 0$ and different values of $\kappa = -0.1, -1, -10$.

dropped down to the present day value of the vacuum energy. It is tempting to identify these gravitationally produced particles with the sources of ultra-high energy cosmic rays.

For the case of initially negative H (contraction regime) the solution in the absence of matter and vacuum energies demonstrates very interesting feature that the Hubble parameter grows up to zero, becomes positive and after reaching some maximum value turns to MD regime. Thus in the "empty" universe cosmological solutions always turn into expanding ones with $H = 2/3t$ even if started from contraction. In this simple case cosmology is non-singular and contraction turns into expansion (see figure 6)).

More interesting and more realistic situation is non-vanishing energy density of matter and/or vacuum. In this case the line $H = 0$ cannot be crossed, at least as we were able to check on some simple examples (see figure 8). We have not found any realistic case of non-singular contraction but cannot exclude that such may exist.

This approach can be also implemented in multidimensional case [9] and in AdS spaces [12]. It is worth to note that a change in number of dimensions does not request a change in the dimensionality of the coefficients in the action.

As any other higher derivative modifications of gravity, the theory introduced here possibly encounters all the problems which such theories are known to have in quantum case [10, 4]. Still more detailed investigation seems necessary.

Acknowledgments

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